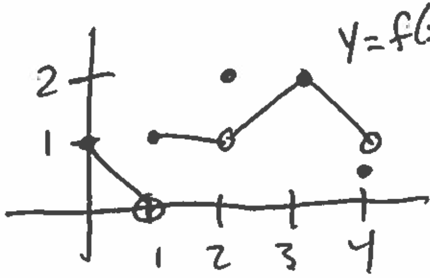
Ex: (2)



x	Left	Right
0	DNE	1
1	0	1
2	1	1
3	2	2
4	1	DNE

Ex: (3) Show $f(x)=\sin(1/x)$ has no limit at $x=0$.

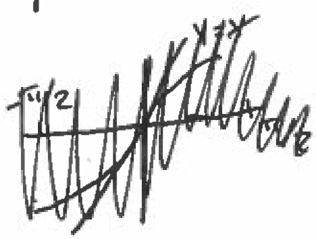
$\sin\left(\frac{1}{x}\right)=1$ whenever $\frac{1}{x}=\frac{\pi}{2}+2\pi n$ for some $n\in\mathbb{Z}$

So when $x=\frac{1}{\pi/2+2\pi n}$. But as $n\to\infty$, $x\to 0$. So when $\varepsilon=1$, no $\delta>0$ will work.

Theorem (7): $\lim\limits_{\theta\to 0}\frac{\sin\theta}{\theta}=1$ Sandwich Theorem between

~~Sandwich theorem,~~

~~betw~~



$\cos\theta<\frac{\sin\theta}{\theta}<1$

when $\theta\in(0,\frac{\pi}{2})$.

Ex: (4) (a) $\lim\limits_{h\to 0}\frac{\cos h-1}{h}=0$; $\frac{\cos h-1}{h}=\frac{1-2\sin^2(h/2)-1}{h}=-\frac{2\sin(h/2)}{h/2}\cdot\sin(h/2)$

So $\lim\limits_{h\to 0}\frac{\cos h-1}{h}=(-1)(0)=0$. (half-angle formula)

(b) $\lim\limits_{x\to 0}\frac{\sin(2x)}{5x}=\lim\limits_{x\to 0}\frac{(2/5)\sin(2x)}{(2/5)5x}=\frac{2}{5}\lim\limits_{x\to 0}\frac{\sin(2x)}{2x}=\frac{2}{5}(1)=\frac{2}{5}$

(c) $\lim\limits_{t\to 0}\frac{\tan t\sec(2t)}{3t}=\lim\limits_{t\to 0}\frac{1}{3}\frac{\sin t}{t}\cdot\frac{1}{\cos t}\cdot\frac{1}{\cos 2t}=\frac{1}{3}(1)(1)(1)=\frac{1}{3}$.

2.5: Continuity

A fcn is continuous at c if $\lim_{x\to c} f(x) = f(c)$

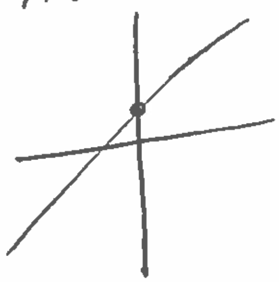
right-cont $\lim_{x\to c^+} f(x) = f(c)$

left-cont $\lim_{x\to c^-} f(x) = f(c)$.

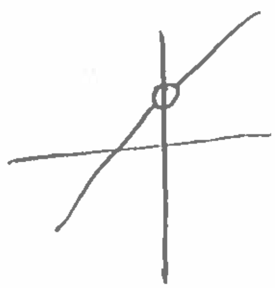
Otherwise we say f is discontinuous at c.

Ex: (1) $f(x) = \lfloor x \rfloor$ is not continuous at $n \in \mathbb{Z}$.

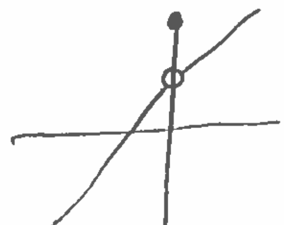
At $x=0$



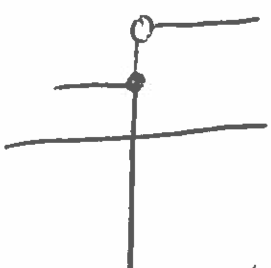
continuous



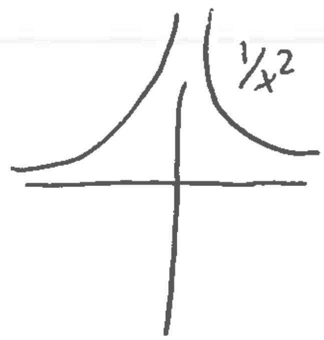
Discontinuous



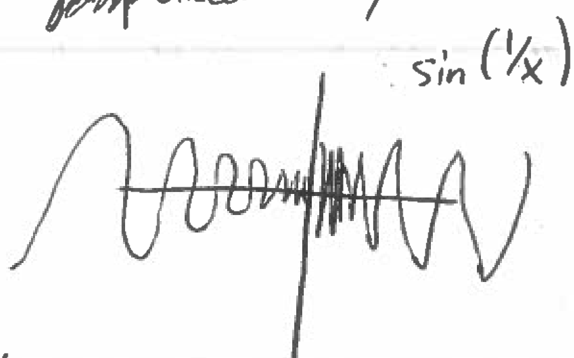
Removable ~~jump~~ discontinuity



jump discontinuity



infinite discontinuity



Oscillating discont.

Def: A fcn which is continuous ~~on its~~ at every point in its domain is called a Continuous fcn.

Properties: Suppose f and g are cont. at c.
Then the following are cont. at c.

(1) $f \pm g$

(2) $k \cdot f$

(3) $f \cdot g$

(4) $\frac{f}{g}$ provided $g(c) \neq 0$

(5) f^n

(6) $f^{1/n}$

Ex: (2) (a) All polynomials are cont. $\lim_{x \to c} P(x) = P(c)$

(b) All rationals are cont. provided $Q(c) \neq 0$, $\frac{P}{Q}$.

(c) $|x|$ is cont. It is a polynomial for $x > 0$ and $x < 0$.
Need only check $x = 0$.

Remarks: (1) If f is cont. and f^{-1} exists then f^{-1} is cont.
(Switch ε and δ)

Theorem: (2) If f is cont. at c and g is cont. at $f(c)$
then ~~fog~~ $g \circ f$ is cont. at c.

Ex: (3) (a) $\sqrt{x^2 - 2x - 5}$ (b) $\frac{x^{2/3}}{1 + x^4}$ (c) $\left|\frac{x-2}{x^2-2}\right|$ (d) $\left|\frac{x \sin x}{x^2+2}\right|$

All cont. on there domain.

__Theorem:__ Suppose g is cont. at b and $\lim_{x \to c} f(x) = b$. Then

$$\lim_{x \to c} g(f(x)) = g\left(\lim_{x \to c} f(x)\right) = g(b).$$

__Proof:__ Let $\varepsilon > 0$. There exists $\delta_1 > 0$ such that

if $0 < |y - b| < \delta_1$ then $|g(y) - g(b)| < \varepsilon$.

Since $\lim_{x \to c} f(x) = b$, there exists $\delta > 0$ such that if

$0 < |x - c| < \delta$ then $|f(x) - b| < \delta_1$.

Setting $y = f(x)$, we have

if $0 < |x - c| < \delta$ then $|y - b| < \delta_1$ then $|g(y) - g(b)| < \varepsilon$.

__Ex:__ (4) (a) $\lim_{x \to \pi/2} \cos\left(2x + \sin\left(\frac{3\pi}{2} + x\right)\right) = \cos\left(\lim_{x \to \pi/2} 2x + \lim_{x \to \pi/2} \sin\left(\frac{3\pi}{2} + x\right)\right)$

$= \cos(\pi + \sin(2\pi)) = \cos(\pi) = -1$

(b) $\lim_{x \to 1} \sin^{-1}\left(\frac{1-x}{1-x^2}\right) = \sin^{-1}\left(\lim_{x \to 1} \frac{1-x}{(1-x)(1+x)}\right) = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}.$

(c) $\lim_{x \to 0} \sqrt{x+1}\, e^{\tan x} = \lim_{x \to 0} \sqrt{x+1} \cdot e^{\lim_{x \to 0} \tan x} = \sqrt{1} \cdot e^0 = 1.$

__Intermediate Value Theorem:__ If f is cont. on $[a, b]$ and y_0 is between $f(a)$ and $f(b)$ then there is some $c \in [a, b]$ such that $f(c) = y_0$.

This is the __connectedness__ property of cont. fcns.

Ex(5): Use the intermediate value theorem to prove

$$\sqrt{2x+5} = 4-x^2 \quad \text{has a solution.}$$

Let $f(x) = \sqrt{2x+5} + x^2$ defined on $[-5/2, \infty)$.

It is cont. $f(0) = \sqrt{5} < 4$

$f(2) = 7 > 4$. So $\exists\, c \in [0,2]$ st $f(c) = 4$.

Ex(6): Show $f(x) = x^3 - x - 1$ has a root between ~~0~~ 1 and ~~4~~?

$f(1) = -1$, $f(2) = 5$, thus $\exists\, c \in [1,2]$ st $f(c) = 0$.

Continuous Extensions:

Let $f(x)$ be a function which is not defined at $x = c$.

If $\lim_{x \to c} f(x) = L$, then $F(x) = \begin{cases} f(x), & x \neq c \\ L, & x = c \end{cases}$ is continuous at c.

Ex(7): Show that $f(x) = \dfrac{x^2+x-6}{x^2-4}$ has a continuous extension at $x=2$.

$$f(x) = \frac{(x-2)(x+3)}{(x-2)(x+2)} = \frac{(x+3)}{x+2} \quad \text{when } x \neq 2.$$

$\lim_{x \to 2} f(x) = \frac{5}{4} = \frac{5}{4}$. So $F(x) = \begin{cases} \frac{x+3}{x+2}, & x \neq \pm 2 \\ \frac{5}{4}, & x = 2 \end{cases}$

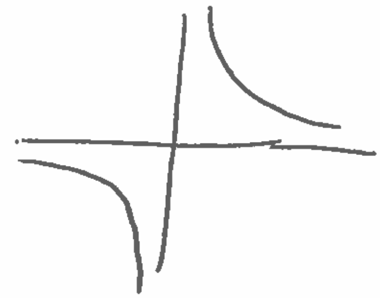
is the continuous extension of f at $x=2$.

2.6: Limits involving infinity, asymptotes of a graph

We now define $\lim_{x\to\infty} f(x) = L$ if $\forall \varepsilon > 0$ $\exists M > 0$ such that if $x > M$ then $|f(x) - L| < \varepsilon$.

Similarly, $\lim_{x\to -\infty} f(x) = L$ if for any $\varepsilon > 0$ there exists $M < 0$ such that if $x < M$ then $|f(x) - L| < \varepsilon$.

Ex(1): (a) $\lim_{x\to\infty} \frac{1}{x} = 0$, $\lim_{x\to -\infty} \frac{1}{x} = 0$.



(a) $M = \frac{1}{\varepsilon}$ or (b) $M = -\frac{1}{\varepsilon}$.

Remark: All the limit laws from before still hold for limits at $\pm\infty$.

Ex(2): (a) $\lim_{x\to\infty} \left(5 + \frac{1}{x}\right) = 5 + \lim_{x\to\infty} \frac{1}{x} = 5$

(b) $\lim_{x\to\infty} \frac{\pi\sqrt{3}}{x^2} = \lim_{x\to\infty} \pi\sqrt{3} \cdot \frac{1}{x} \cdot \frac{1}{x} = 0$

Ex(3): (a) $\lim_{x\to\infty} \frac{5x^2+8x-3}{3x^2+2} = \lim_{x\to\infty} \frac{5x^2+8x-3}{3x^2+2} \cdot \frac{1/x^2}{1/x^2} = \lim_{x\to\infty} \frac{5 + 8/x - 3/x^2}{3 + 2/x^2} = \frac{5}{3}$

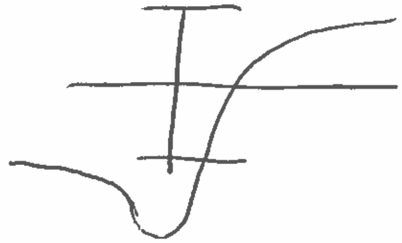
(b) $\lim_{x\to\infty} \frac{11x+2}{2x^3-1} = \lim_{x\to\infty} \frac{11x+2}{2x^3-1} \cdot \frac{1/x^3}{1/x^3} = \lim_{x\to\infty} \frac{11/x^2 + 2/x^3}{2 - 1/x^3} = \frac{0}{2} = 0.$

We call these horizontal asymptotes.

We say the line $y = L$ is a horizontal asymptote.

Ex(4): $f(x)=\frac{x^3-2}{|x|^3+1}$. $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \frac{1-2/x^3}{1+1/x^3} = 1$ hor. asymptote at $y=1$

$\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to\infty} \frac{1-2/x^3}{-1+1/x^3} = -1$ hor. asymptote at $y=-1$



Ex(5): $f(x)=e^x$ at $-\infty$. $M=\ln\varepsilon$.

Ex(6): (a) $\lim\limits_{x\to\infty} \sin(1/x) = \sin\left(\lim\limits_{x\to\infty}\frac{1}{x}\right) = \sin(0) = 0$.

(b) $\lim\limits_{x\to\infty} x\sin(1/x) = \lim\limits_{t\to 0^+} \frac{\sin(t)}{t} = 1$ (Same for $-\infty$).

Ex(7): We can investigate $\lim\limits_{x\to 0^+} f(1/x)$ by examining $\lim\limits_{t\to\infty} f(t)$.

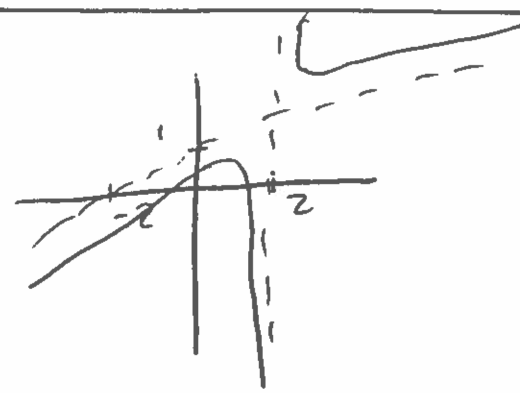
(similarly for $-\infty$).

$\lim\limits_{x\to 0^-} e^{1/x} = \lim\limits_{t\to-\infty} e^t = 0$.

Slant or oblique asymptotes

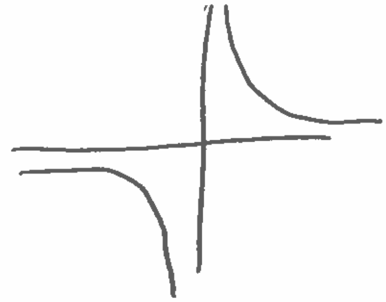
Ex(8): $f(x)=\frac{x^2-3}{2x-4}$

$$\begin{array}{r} \frac{x}{2}+1 \\ 2x-4\,\overline{\big)\,x^2-3} \end{array}$$ + remainder of 1



So $\frac{x^2-3}{2x-4} = \frac{x}{2}+1+\frac{1}{2x-4}$ $\lim\limits_{x\to\infty} = \frac{1}{2x-4} = 0$ so looks like $\frac{x}{2}+1$.

Infinite Limits: $f(x) = \frac{1}{x}$



$\lim_{x \to 0^+} f(x) = \infty$

Def^n: $\lim_{x \to c} f(x) = \infty$ if for $B > 0$ there is some $\delta > 0$ such that

if $|x - c| < \delta$ then $f(x) > B$.

Similarly for $-\infty$ and Right/Left Side Limits.

These are called vertical asymptotes.

We say the line $x = c$ is a vertical asymptote for f.

Ex (9): $\lim_{x \to 0} \frac{1}{x^2} = \infty$, $\delta = \frac{1}{\sqrt{B}}$ $\quad \left(\frac{1}{x^2} > B\right) \Rightarrow |x| < \frac{1}{\sqrt{B}}$.

Ex(10): $y = \frac{x+3}{x+2}$ has vert. asymptote at $x = -2$.

Ex(11): $f(x) = \frac{-8}{x^2 - 4}$ has vert. asymptotes at $x = \pm 2$.

Ex(12): $f(x) = \ln x$ has vert. asymptote at $x = 0$ (right side limit)

Exercise: Find vert. asymptotes for $\tan x$, $\sec x$, $\csc x$.

Remark: For more limit tools, we must first define derivatives!! :)